Charged Particles in Electromagnetic Field in Classical Case

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Abstract

The study of the physical system consisting of charged particles in electromagnetic field constitute a major part of the whole of physics. Here starting with the general laws of classical electrodynamics in the covariant form and then consider special cases of uniform and non–uniform electromagnetic fields with examples to find the trajectories in exact form or in an approximation.

Keywords: *Electromagnetic field, charged particles.*

Introduction

Here we consider the motion of a charged particle in various electromagnetic fields in the absence of a medium. We follow the treatments of Landau-Lifshitz and Jackson here.

The equation of motion of a charge in an electromagnetic field can be written as

$$\frac{dp}{dt} = eE + \frac{e}{c} V * H \qquad \text{……….1.1}$$

The expression on the right of equation (1.1.) is called **Lorentz Force**.

The work done on the charged particle by the electric field is given by

$$\frac{d\varepsilon_{kin}}{dt} = eE.v \qquad \text{……….1.2}$$

The magnetic field does no work on a charge moving in it because the force which the magnetic field exerts is always perpendicular to the velocity of the charge

Hence the energy of a charged particle in a constant time independent electromagnetic field can be written as

$$\varepsilon = \frac{mc^2}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} + e\phi \qquad \text{……….1.3}$$

The presence of the field adds to the energy of the particle i.e. eϕ the potential energy of the charge in the field. Energy depends only scalar but not on the vector potential. This means that the magnetic field does not affect the energy of the charge, only electric field can change the energy.

The covariant equation of motion of a charged particle can be written as :

$$m\frac{d^\alpha X^\alpha}{d\tau^2} = \frac{q}{c}\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\frac{dX_\beta}{d\tau} \qquad \text{……….1.4}$$

Where τ is the proper time

1.1 Motion in Constant Electric and Magnetic Fields

Consider the motion of a charged particle 'e' moving in a combination of electric and magnetic fields E and H, both uniform and constant. For this the equation of motion along the direction of H as z – axis will be

$$m\dot{v} = eE + \frac{e}{c} V * H$$

……….1.11

$$m\ddot{x} = \frac{e}{c}\dot{y}H$$

or $$m\ddot{y} = eEy\frac{e}{c}\dot{x}H$$

$$m\ddot{z} = eEz$$

……….1.12

From the third equation, it is noted that the charge moves with uniform acceleration in the z-direction and is given by

$$Z = \frac{eE_z}{2m}t^2 + \upsilon_{0z}t$$

……….1.13

Multiplying the 2nd equation by i and combining with the first, we get

$$\frac{d}{dt}\left(\dot{x} + i\dot{y}\right) + i\omega\left(\dot{x} + i\dot{y}\right) = i\frac{e}{m}E_y$$

or

$$\dot{x} + i\dot{y} = \alpha e^{-i\omega t} + \frac{cE_y}{H}$$

……….1.14

Separating the real and imaginary parts we get

$$\dot{x} = a\cos\omega t + \frac{cE_y}{H}$$

$$\dot{y} = -\alpha\sin\omega t$$

……….1.15

The average velocity of the particle along x-axis and y-axis are

$$\overline{\dot{x}} = \frac{cE_y}{H}, \overline{\dot{y}} = 0$$

……….1.16

Integrating and choosing the constant of integration so that at t= 0, x = y = 0 and we get

$$x = \frac{\alpha}{\omega}\sin\omega t + \frac{c\varepsilon_y}{H}t$$

$$y = \frac{\alpha}{\omega}\left(\cos\omega t - 1\right)$$

……….1.17

These equations define a **trochoid**. Depending on wheather a is large or smaller in absolute value than the quantity $< \dot{z} > \approx \upsilon_0 + \omega_H(x) \approx \frac{\upsilon^2}{\omega_H R}$ the projection of the trajectory on the plane xy.

If $\alpha = \frac{-c\varepsilon_y}{H}$,then

$$x = \frac{c\varepsilon_y}{\omega H}\left(\omega t - \sin\omega t\right)$$

……….1.18

$$y = \frac{cE_y}{\omega H}\left(1 - \cos\omega t\right)$$

……….1.19

These gives the projection of the trajectory on the xy plane is a **cycloid.**

All the above formulas are valid for the velocity of the particle is small compared with the velocity of light and electric and magnetic fields satisfy the condition that

$$\frac{E_y}{H} << 1$$

……….1.20

Examples:

1) Electric and Magnetic Field are Parallel:

To calculate the relativistic motion of a charged particle in parallel uniform electric and magnetic fields. In this case the magnetic field has no influence on the motion along the common direction of E and H (along the z-axis) and hence only the influence of electric field.

So the equation of motion in the xy-plane will be

$$\dot{P}_x = \frac{e}{c} H \upsilon_y , \dot{P}_y = \frac{-e}{c} H \upsilon_x$$

……….1.21

$$z = \frac{\varepsilon_0}{eE} \cosh \frac{E}{H} \phi$$

……….1.22

This gives the motion of the charged particle in **parametric form** and the trajectory is a helix with radius $\frac{cpt}{eH}$ and monotonically increasing step, along which the particle moves with decreasing angular velocity $\dot{\phi} = \frac{eHc}{\varepsilon_{kin}}$ with a velocity along the z-axis which tends toward the value c.

II) Electric and Magnetic Field are Mutually Perpendicular:

For this, the equation of motion for the charged particle in which H is along z-direction and E along y-direction and E = H will be

$$\frac{dp_x}{dt} = \frac{e}{c} E \upsilon_y , \frac{dp_y}{dt} = eE\left(1 - \frac{\upsilon_y}{c}\right)$$

$$\frac{dp_z}{dt} = 0$$

……….1.23

which gives P_z = constant

It gives the motion of the particle in **parametric form** (i.e. parameter P_y) where the velocity increases most rapidly in the direction perpendicular to E and H along X-axis.

1.2 Motion in Non-Uniform, Static Magnetic Fields

Let us consider a non-uniform static magnetic field which varies slowly with distance in such a manner that the usual perturbation theory can be applied to get approximate solutions. For this the distance over which $\vec{H}$ changes appreciably in magnitude or direction must be much greater than the gyration radius of the particle.

As an example consider a magnetic field which is independent of z. In the X-Y plane the lines of force are not parallel but slightly curved with a radius of curvature R that is large compared with the gyration radius a. Due to the symmetry of the problem it is advantageous to use cylindrical co-ordinate (ρ, ϕ, Z) with the origin at the centre of curvature.

The magnetic induction depends on the ratio $\frac{R}{\rho}$ and has only the ϕ component

$$H_\phi = H_0 \left(\frac{R}{\rho} \right)$$

……….1.24

The Lorentz force equation

$$m\ddot{\vec{r}} = e\left(\vec{E} + \vec{\upsilon} * \vec{H}\right)$$

……….1.25

becomes in cylindrical coordinates for the above magnetic field.

$$\ddot{\rho} - \rho\dot{\phi}^2 = -\omega_H \frac{R}{\rho}\dot{z}$$
……….1.26

$$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0$$
……….1.27

$$\ddot{z} = \omega_H \ln\left(\frac{R}{\rho}\right) + \upsilon_0$$
……….1.28

The equation (1.27) can be written as

$$\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = 0$$
……….1.29

We obtain $\rho^2 \dot{\phi} = a$ constant which we write as $R\upsilon_{\square}$

Had the magnetic field been a constant the trajectory would have been a helix, since H is not uniform, but does not change drastically we expect that ρ would be have a value close to R, when the trajectory of the helix has a radius small compared to R. So ρ can be put as ρ=R+x and $f\left(\frac{\rho}{R}\right)$ can be expanded in powers of $\frac{x}{R}$, appropriately with the approximation $\dot{z} \cong \omega_H x + \upsilon_0$ the radial equation of motion is approximately given by

$$\ddot{x} + \left(\omega_H^2 + \frac{3}{R^2}\upsilon_{\square}^2\right)x \approx \frac{\upsilon_{\square}^2}{R} - \omega_H\upsilon_0$$
……….1.30

This is the equation of motion of a harmonic oscillator around x with a displaced equilibrium position

$$< x > \cong \frac{\upsilon_{\square}^2}{\omega_H{}^2 R} - \frac{\upsilon_0}{\omega_H}$$
……….1.31

Here we have assumed $\upsilon_H << \omega_H R$. The mean value of $\dot{z}$ is

$$< \dot{z} > \approx \upsilon_0 + \omega_H(x) \approx \frac{\upsilon_{\square}^2}{\omega_H R}$$
……….1.32

This is known as **Curvature drift**. If the spatial variation of the magnetic field is such that the gradient of the field is perpendicular to the direction of $\vec{H}$. then an analysis analogous to the above gives a gradient drift to velocity. Both then drifts are trouble some in confining high temperature plasmas, and the twisted figure eight toroidal design is made to keep the **plasma confined**.

Conclusion

The above considerations are used in various ways such as cathode ray oscilloscopes and tubes, cyclotrons and other accelerators, motion of charged particles in the ionosphere, synchro-cyclotron (Relativistic ion Cyclotron), $\frac{e}{m}$ of an electron by Thomson method, Thomson mass spectrograph, Aston's mass Spectrograph, Dempster, Mass Spectrograph, Magnetron Betatron, Hall effects etc.

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